Gli integrali indefiniti

$y = x^2 \Rightarrow y' = 2x$

Se due funzioni hanno la stessa derivata in un intervallo, differiscono per una costante

$y' = 2x \Rightarrow y = x^2 + C$

$y = x^2$ **si dice primitiva di** $y = 2x$

In generale, $y = F(x)$ **si dice primitiva di** $y = f(x)$ **se** $F'(x) = f(x)$

$\int 2x\,dx = x^2 + C$

$2x$ **si chiama funzione integranda.**

$\int f(x)\,dx$ **indica l'insieme delle funzioni che hanno per derivata** $f(x)$ **, ed è uguale ad una qualunque primitiva di** $f(x)$ **, a cui sommeremo una costante arbitraria.**

In generale, $\int f(x)\,dx = F(x) + C$

Qualche regola di integrazione:

0) $\int dx = x + C$

1) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (\ n \neq -1\ )$

2) $\int \frac{1}{x}\,dx = \ln|x| + C$

3) $\int sen\,x\,dx = -\cos x + C$

4) $\int \cos x\,dx = sen\,x + C$

5) $\int e^x\,dx = e^x + C$

Es.1) $\int x^3\,dx = \frac{x^4}{4} + C$

Es.2) $\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$

Es.3) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \sqrt{x^3} \cdot \frac{2}{3} + C = \frac{2}{3} x \sqrt{x} + C$

Alcune regole:

1) $\int [f(x) \pm g(x)]\,dx = \int f(x) \pm \int g(x)\,dx$

Es.

$$\int (x^2+senx)\,dx=\int x^2\,dx+\int senx\,dx=\frac{x^3}{3}+(-cosx)+C=\frac{x^3}{3}-cosx+C$$

2) $\int [k\cdot f(x)]\,dx=k\cdot\int f(x)\,dx$ (con k costante)

Es. $\int 5x^6\,dx=5\int x^6\,dx=5\cdot\frac{x^7}{7}+C=\frac{5}{7}x^7+C$

Es. $\int (5x^2-3x+7)\,dx=\int 5x^2\,dx-\int 3x\,dx+\int 7\,dx=$

$$=5\int x^2\,dx-3\int x^1\,dx+7\int 1\cdot\,dx=\frac{x^2}{2}+7x+C=\frac{5}{3}x^3-\frac{3}{2}x^2+7x+C$$

$$=\frac{5}{3}x^3-\frac{3}{2}x^2+7x+C$$

Integrali di funzioni composte

Premessa

Es. $y=sen(2x)$, ovvero $\begin{cases} t=2x \\ y=sent \end{cases}$

$$y'=\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=D(sent)\cdot D(2x)=\cos t\cdot 2=2\cos(2x)$$

$$D(senf(x))=f'(x)\cdot\cos f(x)$$

Quindi, generalizzando, $\int \cos x\,dx=sen\,x+C$ diventa

$$\int f'(x)\cdot\cos f(x)\,dx=sen\,f(x)+C$$

Formule generalizzate:

1) $\int f'(x)[f(x)]^n\,dx=\frac{[f(x)]^{n+1}}{n+1}+C$ ($n\neq-1$)

2) $\int \frac{f'(x)}{f(x)}\,dx=\ln|f(x)|+C$

3) $\int f'(x)\cdot sen\,f(x)\,dx=-\cos f(x)+C$

4) $\int f'(x)\cdot\cos f(x)\,dx=sen\,f(x)+C$

5) $\int f'(x)\cdot e^{f(x)}\,dx=e^{f(x)}+C$

Tutte le x delle formule degli integrali immediati diventano f(x), tranne il dx, e la funzione integranda originale è moltiplicata per f'(x) (in altre parole, x diventa f(x), e dx diventa f'(x)dx).

Es: $\int 2\cos 2x\,dx=sen\,2x+C$

Es. $\int \cos 2x\,dx=\int \frac{1}{2}\cdot 2\cdot\cos 2x\,dx=\frac{1}{2}\int 2\cdot\cos 2x\,dx=\frac{1}{2}sen\,2x+C$

Es. $\int sen\,\omega t\,dt = \int \frac{1}{\omega}\cdot\omega\cdot sen\,\omega t\,dt = \frac{1}{\omega}\int \omega\cdot sen\,\omega t\,dt = -\frac{1}{\omega}\cos\omega t + C$

Es. $\int \frac{2x+1}{x^2+x-3}dx = \ln\left|x^2+x-3\right| + C$

Es. $\int (2x+1)^2 dx = \int \frac{1}{2}\cdot 2(2x+1)^2 dx = \frac{1}{2}\cdot\int 2(2x+1)^2 dx =$

$= \frac{1}{2}\cdot\frac{(2x+1)^{2+1}}{3} + C = \frac{(2x+1)^3}{6} + C$

$\int tg\,x\,dx = \int \frac{senx}{cosx}dx = \int -\frac{-senx}{cosx}dx = -\int \frac{-senx}{cosx}dx = -\ln|\cos x| + C$

$\int \frac{1}{(x-2)\ln(x-2)}dx = \int \frac{\frac{1}{x-2}}{\ln(x-2)}dx = \ln|\ln(x-2)| + C$

$\int sen\,x\cos x\,dx = \int \cos x\,(senx)^1 dx = \frac{sen^2 x}{2} + C$

6) $\int \frac{x^3}{x^4+1}dx = \frac{1}{4}\int \frac{4x^3}{x^4+1}dx = \frac{1}{4}\ln\left|x^4+1\right| + C = \ln\sqrt[4]{\left|x^4+1\right|} + C$

Altra formula:

$D(arctg\,x) = \frac{1}{x^2+1}$

Quindi $\int \frac{1}{x^2+1}dx = arctg\,x + C$ **(immediato), e**

$\int \frac{f'(x)}{[f(x)]^2+1}dx = arctg\,f(x) + C$ **(generalizzato)**

7) $\int \frac{x}{x^4+1}dx = \frac{1}{2}\int \frac{2x}{(x^2)^2+1}dx = \frac{1}{2}arctg\,x^2 + C$

8) $\int \frac{1}{x\ln x}dx = \int \frac{1}{x}\cdot\frac{1}{\ln x}dx = \int \frac{\frac{1}{x}}{\ln x}dx = \ln|\ln x| + C$

9)

$$\int sen\,x\cos^4 x\,dx = -\int -senx(\cos x)^4\,dx = -\frac{(\cos x)^5}{5} + C = -\cos^5\frac{x}{5} + C$$

10) $\int sen^4 x\cos^3 x\,dx = \int sen^4 x\cos^2 x\cos x\,dx =$

$$= \int sen^4 x(1 - sen^2 x)\cos x\,dx = \int (sen^4 x - sen^6 x)\cos x\,dx =$$

$$= \int (sen^4 x\,cosx - sen^6 x\cos x)\,dx =$$

$$= \int sen^4 x\cos x\,dx - \int sen^6 x\cos x\,dx = \frac{sen^5 x}{5} - \frac{sen^7 x}{7} + C$$

$$= \int sen^4 x(1 - sen^2 x)\cos x\,dx$$

N.B. $\cos^2 x = 1 - sen^2 x$;

$\cos^4 x = (\cos^2 x)^2 = (1 - sen^2 x)^2$;

$\cos^6 x = (\cos^2 x)^3 = (1 - sen^2 x)^3$ …

$$\int sen^7 x\cos^3 x\,dx = \int sen^7 x\cos^2 x\cos x\,dx =$$

$$= \int sen^7 x(1 - sen^2 x)\cos x\,dx = \ldots.$$

$$\int sen^3 x\,dx = \int sen^2 x\,sen\,x\,dx = \int (1 - \cos^2 x)\,sen\,x\,dx = \ldots$$

$$\int \frac{1}{(1+x^2)\,arctg\,x}\,dx = \int \frac{\frac{1}{1+x^2}}{arctg\,x}\,dx = \ln|arctg\,x| + C$$

Es. pag. 1907 n.271

$$\int \frac{2\cos x + sen\,2x}{\cos x}\,dx = \int \frac{2\cos x}{\cos x}\,dx + \int \frac{sen\,2x}{\cos x}\,dx =$$

$$= \int 2\,dx + \int \frac{2\,sen\,x\cos x}{\cos x}\,dx = 2\int dx + \int 2\,sen\,x\,dx = 2x + 2\int sen\,x\,dx =$$

$$= 2x + 2(-\cos x) + C = 2x - 2\cos x + C$$

Oppure:

$$\int \frac{2\cos x + sen\,2x}{\cos x}\,dx = \int \frac{2\cos x + 2\,senx\cos x}{\cos x}\,dx =$$

$$= \int \frac{2\cos x(1 + senx)}{\cos x}\,dx = \int 2(1 + senx)\,dx \ldots$$

Es. pag. 1907 n.277

$$\int \frac{\cos x}{2 - \cos^2 x}\,dx = \int \frac{\cos x}{2 - (1 - sen^2 x)}\,dx = \int \frac{\cos x}{1 + sen^2 x}\,dx =$$

$$=\int \frac{\cos x}{(sen\,x)^2+1}dx=arctg\,sen\,x+C$$

Es. pag. 1907 n. 81

$$\int \frac{e^{x+1}}{3+e^x}dx=\int \frac{e^x\cdot e^1}{3+e^x}dx=e\int \frac{e^x}{3+e^x}dx=e\cdot \ln\left|3+e^x\right|+C=$$

$$=e\cdot \ln\left(3+e^x\right)+C$$

$$\int \frac{1}{(x-2)\ln(x-2)}dx=\int \frac{\frac{1}{x-2}}{\ln(x-2)}dx=\ln\left|\ln(x-2)\right|+C$$

Es. pag. 907 n.289

$$\int \frac{x}{\sqrt{x^2-9}}dx=\int \frac{x}{(x^2-9)^{\frac{1}{2}}}dx=\int x\cdot(x^2-9)^{-\frac{1}{2}}dx=$$

$$=\frac{1}{2}\int 2x\cdot(x^2-9)^{-\frac{1}{2}}dx=\frac{1}{2}\cdot\frac{(x^2-9)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C=\frac{\frac{1}{2}\cdot(x^2-9)^{\frac{1}{2}}}{\frac{1}{2}}+C=$$

$$=(x^2-9)^{\frac{1}{2}}+C=\sqrt{x^2-9}+C$$

Es. pag.1908 n. 315

$$\int \frac{1}{(x-2)\ln(x-2)}dx=\int \frac{\frac{1}{x-2}}{\ln(x-2)}dx=\ln\left|\ln(x-2)\right|+C$$